σ = standard deviation of size distribution in ln units
t = time
W = weight fraction attrited
x = particle diameter in microns
x_f = diameter of fine particles
x_t = diameter of primary particle after attriting for time t
84.13, 50 = subscripts, percentage levels of cumulative size distribution

LITERATURE CITED

1. Forsythe, W. L., Jr., and W. R. Hertwig, *Ind. Eng. Chem.*, 41, 1200 (1949).

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An Approximate Solution for Countercurrent Heat Exchangers

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An approximate solution is presented for the countercurrent parallel plate exchanger with laminar flow. With the use of the integral method, the problem is reduced to one of solving a pair of first-order differential equations in a straightforward manner. Comparisons between the results of this work and those obtained from a more elaborate orthogonal expansion technique are found to be excellent.

Counterflow heat exchangers are extensively used for a variety of applications. Nevertheless, detailed analytical studies of this type of operation have been absent until very recent years. A finite-difference solution on countercurrent mass transfer was first given by King (*3*). Nunge and Gill (*7, 8, 9*), and Stein (*10, 11*) independently obtained an analytical solution for laminar counterflow heat exchangers (parallel plate as well as double pipe). These authors succeeded in generalizing the classical Graetz solution for single stream to the case of two streams in counterflow. The analysis requires the use of denumerable sets of both positive and negative eigenvalues. This approach has been considered to be mathematically correct by Blanco (*1*) who based his argument on the earlier work of Mason (*5, 6*). A check of the references indicate that the problem considered by Mason (*6*) and the counterflow problem are similar but not identical, except for the special case of negligible interphase resistance, that is, $K_w = 0$. Consequently, one cannot accept the validity on the use of this orthogonal expansion technique with absolute certainty to cases where interphase resistance is significant.

In terms of computation, the Nunge-Gill approach requires the solution of an infinite matrix for the evaluation of expansion coefficients. Their suggested method, however, was found unsatisfactory by Stein (*12*) who attempted the solution of the plug flow case by using the Nunge-Gill procedure. Instead, Stein offers an alternative method (named *Argonne Procedure*) for the calculation of expansion coefficients. In a recent note, Blanco, Gill, and Nunge (*2*) pointed out that the difficulty involved in the evaluation of expansion coefficient using the so-called "Nunge-Gill procedure," is only limited to the case of plug flow. For the general case of laminar flow, both procedures yield essentially the same results. Although it is entirely likely that one particular procedure may be more suitable for certain cases, no theoretical argument has been advanced by these authors, which enables the selection of a proper procedure for a given application.

In addition to the proper choice of a numerical procedure for the computation of expansion coefficients, there also exist some other limitations on the use of this orthogonal expansion technique for the detailed study of countercurrent heat exchangers as pointed out elsewhere (*1*). First, like the Graetz problem, the series expansions are slow in convergence for small longitudinal distances and require more terms than can be accurately calculated. Also, as the longitudinal distance increases, due to the rapid increases in values of the exponential function associated with the negative eigenvalues, increasingly fewer terms can be included in the equations from which the expansion coefficient can be calculated. On the other hand, because of the large number of physical parameters involved in the counterflow problem, it is impractical to make calculations for all cases of interest and obtain a so-called "master plot". Instead, calculation has to be made for each specific application. Because of these considerations, it seems desirable to consider an alternative approach to this problem in spite of the theoretical elegance of the Nunge-Gill method.

The object of our work is to present an alternative method to the solution of countercurrent heat exchangers, although the result can be readily adapted to mass transfer apparatus such as hemodialyzer. The approach consists of the use of the integral approximation which reduces the energy equations into a pair of first-order differential equations. The use of the integral approximation eliminates one independent variable (the transverse direction across the conduit) by direct consideration. Furthermore, if one treats the length of the exchanger as a dependent variable and considers the problem as one in which the size of the exchanger is to be determined for a given heat transfer requirement, then the two first-order equations can be numerically integrated in a straightforward manner, thus circumventing the difficulty of matching the inlet condition for both streams at the opposite ends of the exchanger, as required by the previous investigators. The integral approach has been found satisfactory in the solution of cocurent hemodialyzer (*13*). Apparently, some of the earlier investigators had considered the possibility of employing an integral method for the solution of counterflow exchangers (*4, 5*); however, for some unspecified reason, this approach was not fully explored.

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ANALYSIS

A schematic description of the counterflow parallel plate heat exchanger is given in Figure 1. For convenience of analysis, the heat exchanger is assumed to be consisting of three parallel plates placed at distances of h_1 and h_2, respectively. Two streams designated as *stream 1* and *stream 2* flow in opposite directions to one another, and the two outer plates are assumed to be perfectly insulated. This same physical description is equally valid for a multistream exchanger, provided that the heating (or cooling) is symmetrical and h_1 (as well as h_2) is interpreted as half width of the individual passage. Two separate coordinate systems (x_1, y_1 and x_2, y_2) are used for the respective streams and they are related by $x_1 + x_2 = L$, where L is the physical length of the exchanger.

If we assume that the initial temperature for either stream is uniform, and, that the conventional boundary layer analysis is applicable to the present problem, the integral energy equation for either stream can be written as

$$\frac{d}{dx_j}\int_0^{\Delta_j}[(T_j)_i - T_j]\,u_j\,dy_j = \alpha_j\left(\frac{\partial T_j}{\partial y_j}\right)_0 \qquad (1)$$

$$j = 1, 2$$

where Δ_j is the thermal boundary layer thickness. The initial, boundary and interphase conditions are given as

$$T_j = (T_j)_i \quad \text{at } x_j = 0 \qquad (2)$$

$$\frac{\partial T_j}{\partial y_j} = 0 \quad \text{at } y_j = h_j \qquad (3)$$

$$-k_1\left(\frac{\partial T_1}{\partial y_1}\right) = k_2\left(\frac{\partial T_2}{\partial y_2}\right) = K(T_2 - T_1)$$

$$\text{at } y_1 = y_2 = 0 \qquad (4)$$

where K is the reciprocal of the overall resistance to heat conduction across the interphase and is equal to the thermal conductivity of the wall divided by its thickness. The velocity profiles of both streams are assumed to be fully developed and are given by the familiar parabolic distributions.

$$u_j = 6\bar{U}_j\left[\frac{y_j}{h_j} - \left(\frac{y_j}{h_j}\right)^2\right], \quad j = 1, 2 \qquad (4a)$$

It should be pointed out that unlike the orthogonal expansion technique, the present method is not restricted to the case of fully developed flow. The effect of developing velocity profile for either or both streams can be easily considered. For such cases, in addition to Equation (1), momentum equations will be needed.

As is customarily done for integral solutions, temperature profiles for both streams will be assumed and substituted into Equation (1). For the present problem, the assumed temperature profile will take different expressions depending upon whether the temperature profile is already fully established or not (or more properly, whether $\Delta_j < h_j$ or $\Delta_j = h_j$). For a given exchanger length, the thermal boundary layer in either passage could be in the developing stage of being fully developed. Consequently, a number of situations could arise depending upon the length of the exchanger as well as the relative boundary layer growth rate of the streams. All possible combinations are illustrated in Figure 2.

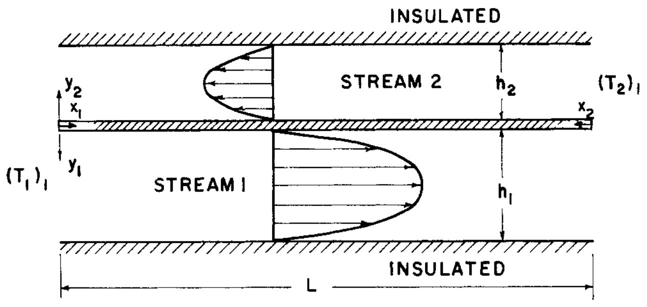

Fig. 1. Schematic diagram of counter flow parallel plate heat exchanger.

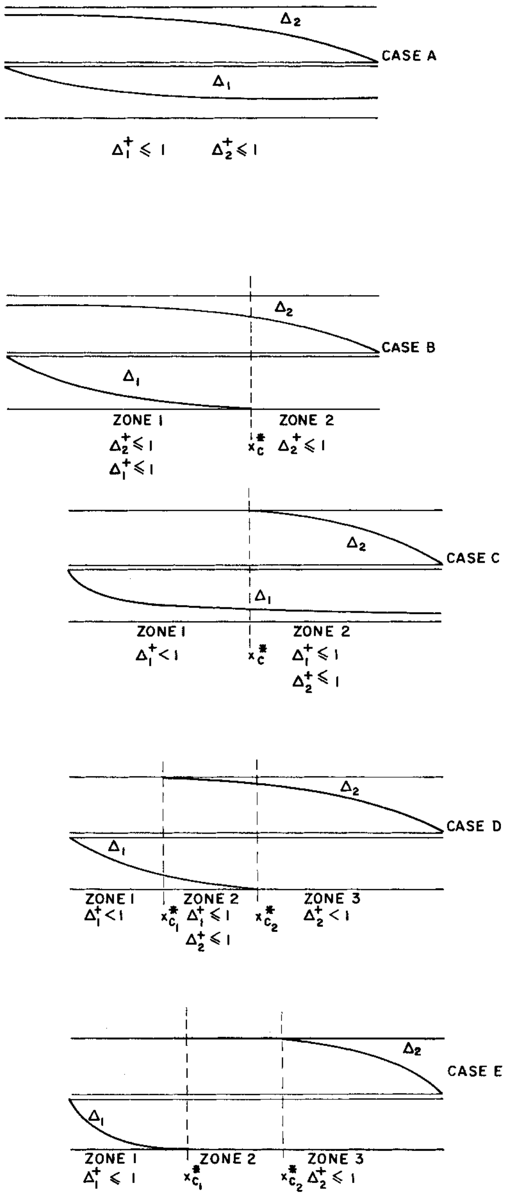

Fig. 2. Five different cases which may be encountered in counterflow heat exchanger.

Case (A)

This is the case where the thermal boundary layers for either stream has not become fully developed. First, a dimensionless temperature is introduced. Let

$$\theta_j = \frac{T_j - (T_1)_i}{(T_2)_i - (T_1)_i}, \quad j = 1, 2 \qquad (5)$$

The inlet conditions of Equation (2) become

$$\theta_1 = 0, \quad x_1 = 0 \qquad (6)$$

$$\theta_2 = 1, \quad x_2 = 0 \qquad (7)$$

The following temperature profiles are assumed:

$$\theta_1 = b_1\left[1-\frac{3}{2}\frac{y_1}{\Delta_1}+\frac{1}{2}\left(\frac{y_1}{\Delta_1}\right)^3\right] \quad (8)$$

$$\theta_2 = a_1-\frac{3}{2}(a_1-1)\frac{y_2}{\Delta_2}+\left(\frac{a_1-1}{2}\right)\left(\frac{y_2}{\Delta_2}\right)^3 \quad (9)$$

Equations (8) and (9) are so selected that the compatibility conditions, $\theta_1 = 0$, $\theta_2 = 1$, $\frac{\partial \theta_j}{\partial y_j} = 0$ at $y_j = \Delta_j$, are satisfied. Unlike the integral solution for single stream problems, there are two unknown quantities (b_1, Δ_1 and a_1, Δ_2) present in each of the temperature expressions.

Substituting Equations (8), (9), and (4a) into Equation (1) and carrying out some routine rearrangement, the following dimensionless expressions are obtained:

$$\frac{d}{dx_1^*}\left[\left(\frac{1}{10}\Delta_1^{+2}-\frac{1}{24}\Delta_1^{+3}\right)b_1\right]=\frac{1}{2}\frac{b_1}{\Delta_1^+} \quad (10)$$

$$-\frac{d}{dx_1^*}\left[\left(\frac{1}{10}\Delta_2^{+2}-\frac{1}{24}\Delta_2^{+3}\right)(1-a_1)\right] = \left(\frac{1}{2}\right)\frac{1-a_1}{\Delta_2^+}\frac{1}{K_N}\frac{1}{H} \quad (11)$$

where

$$H = \frac{W_2\,C_{p2}}{W_1\,C_{p1}} \quad (12a)$$

$$K_N = \frac{h_2\,k_1}{h_1\,k_2} \quad (12b)$$

$$x_1^* = \frac{x_1}{h_1(N_{Pe})_1} \quad (12c)$$

W_j, the mass velocity, is defined as $\overline{U}_j h_j \rho_j$, $j = 1, 2$.

$$(N_{Pe})_1 = (N_{Re})_1(N_{Pr})_1 = \left(\frac{2h_1\overline{U}_1\rho_1}{\mu_1}\right)\left(\frac{C_{p1}\mu_1}{K_1}\right) \quad (12d)$$

$$\Delta_j^+ = \frac{\Delta_j}{h_j}, \quad j = 1, 2 \quad (12e)$$

From Equation (4), the interphase condition is given as

$$\frac{b_1}{\Delta_1^+} = \frac{1}{K_N}\frac{1-a_1}{\Delta_2^+} = \left(\frac{2}{3}\right)\frac{a_1-b_1}{K_W} \quad (13)$$

where

$$K_W = \frac{k_1}{h_1 K} \quad (13a)$$

The explicit relationship between the thermal boundary layer thickness and the interphase temperature can be deduced directly from Equation (13) as

$$\Delta_1^+ = \frac{3}{2}K_W\frac{b_1}{a_1-b_1} \quad (14a)$$

$$\Delta_2^+ = \frac{3}{2}\frac{K_W}{K_N}\frac{1-a_1}{a_1-b_1} \quad (14b)$$

Substituting Equations (14a) and (14b) into Equations (10) and (11) and carrying out the differentiation, the following pair of first-order differential equations are obtained:

$$f_1\frac{da_1}{dx_1^*}+f_2\frac{db_1}{dx_1^*}=f_3 \quad (15)$$

$$g_1\frac{da_1}{dx_1^*}+g_2\frac{db_1}{dx_1^*}=-g_3 \quad (16)$$

where

$$f_1 = -\frac{2b_1^3}{(a_1-b_1)^3}\left[1-\frac{15}{16}K_W\frac{b_1}{a_1-b_1}\right] \quad (17a)$$

$$f_2 = \frac{b_1^2}{(a_1-b_1)^2}\left[\frac{3a_1-b_1}{a_1-b_1}-\frac{5}{8}K_W\frac{b_1(4a_1-b_1)}{(a_1-b_1)^2}\right] \quad (17b)$$

$$f_3 = \frac{40}{27}\frac{a_1-b_1}{K_W^3} \quad (17c)$$

$$g_1 = -\frac{(1-a_1)^2}{(a_1-b_1)^2}\left[\frac{a_1-3b_1+2}{a_1-b_1} - \frac{5}{8}\frac{K_W}{K_N}\frac{(1-a_1)(a_1-4b_1+3)}{(a_1-b_1)^2}\right] \quad (18a)$$

$$g_2 = \frac{2(1-a_1)^3}{(a_1-b_1)^3}\left[1-\frac{15}{16}\frac{K_W}{K_N}\frac{1-a_1}{a_1-b_1}\right] \quad (18b)$$

$$g_3 = \frac{40}{27}\frac{K_N^2}{K_W^3}\frac{1}{H}(a_1-b_1) \quad (18c)$$

From Equations (15) and (16), the following expressions are obtained:

$$\frac{da_1}{dx_1^*} = \frac{f_3g_2+f_2g_3}{f_1g_2-f_2g_1} \quad (19)$$

$$\frac{db_1}{dx_1^*} = -\frac{(f_1g_3+f_3g_1)}{f_1g_2-f_2g_1} \quad (20)$$

Equations (19) and (20) provide the basis for numerical computation. The computation, however, cannot begin at $x_1^* = 0$ because of its being a singular point. Instead, the calculation will start at an arbitrarily small distance from $x_1^* = 0$. For this reason, an asymptotic expression for b_1 for small value of x_1^* is required, and this can be obtained in the following manner. First, by combining Equations (14a) and (10), one has

$$\frac{d}{dx_1^*}\left[\frac{b_1^3}{(a_1-b_1)^2}-\frac{5}{8}K_W\frac{b_1^4}{(a_1-b_1)^3}\right]=\frac{40}{27}\frac{a_1-b_1}{K_W^3} \quad (21)$$

for sufficiently small values of x_1^*, $b_1 \simeq 0$; consequently, only the first term on the left hand side of Equation (21) need be considered. Also, at $x_1^* \simeq 0$,

$$a_1 \simeq a_{1,0}$$

$$a_1 - b_1 \simeq a_{1,0}$$

Equation (21) becomes

$$\frac{d}{dx_1^*}\left(\frac{b_1^3}{a_{1,0}^2}\right) \simeq \left(\frac{40}{27}\right)\frac{a_{1,0}}{K_W^3} \quad (22)$$

Simple integration of Equation (22) yields the desired result, or

$$b_1 \simeq \left(\frac{40}{27}\right)^{1/3}\frac{a_{1,0}}{K_W}(x_1^*)^{1/3} \quad (23)$$

The computation scheme can be described as follows: First, the system parameters H, K_N, and K_W are specified. A proper value of a_1 at an arbitrary small value of x_1^* (say $x_1^* = 10^{-4}$) is assumed (that is, $a_{1,0}$). The corresponding value of b_1 can be readily computed from Equation (23). With the initial conditions fully specified, Equations (19) and (20) can be integrated numerically in a straightforward manner until a_1 equals unity. A relationship of L^* vs. $a_{1,0}$ is thus obtained. Physically speaking, the procedure is equivalent to the determination of the size of the heat exchanger for a given heat transfer requirement. In this connection, it should be realized that the choice of the value of $a_{1,0}$ is not entirely arbitrary. For the present case, the thermal boundary layer for both streams are developing. The assumed value of $a_{1,0}$ should be compatible with the condition, $\Delta_2^+ \leq 1$. In other words, $a_{1,0}$ shall always be within the following limits:

$$\frac{1}{1+\frac{2}{3}\frac{K_N}{K_W}} < a_{1,0} < 1 \tag{24}$$

Case (B)

The thermal boundary layer for stream 2 is developing throughout the entire length of the exchanger while that of stream 1 becomes fully developed at $x_1^* = x^* < L^*$. For convenience of analysis, the entire length of the exchanger can be divided into two parts. These are:

Zone 1. $x_1^* < x_c^*$, $\Delta_1^+ \leq 1$ and $\Delta_2^+ \leq 1$

Zone 2. $x_1^* > x_c^*$, $\Delta_1^+ = 1$, $\Delta_2^+ < 1$

For Zone 2, a different temperature profile will be used. This is given as:

$$\theta_1 = b_1 + b_2\left[\frac{y}{h_1} - \frac{1}{3}\left(\frac{y}{h_1}\right)^3\right] \tag{25}$$

Equation (25) satisfies the condition of perfect insulation at the outer wall (that is, $y_1 = h_1$). The relationship between b_1 and b_2, can be found by using the interphase condition.

The interphase condition of Equation (4) with θ_1 given by Equation (25) and θ_2 given by Equation (9) becomes,

$$-b_2 = -\frac{3}{2}\frac{1}{K_N}\frac{a_1 - 1}{\Delta_2^+} = \frac{1}{K_W}(a_1 - b_1) \tag{26}$$

Equation (1) now has to be modified to account for the fact that the thermal boundary layer is fully established. By a simple analysis, the correct expression is given as

$$\frac{d}{dx_j}\int^{h_j} \Gamma_j u_j dy_j = -\alpha_j\left(\frac{\partial T_j}{\partial y_j}\right)_0, \quad j = 1, 2 \tag{1a}$$

The integrated energy equation for stream 1 can be obtained by substituting Equations (25) and (4a) into Equation (1a). The integration is now carried out from $y_1 = 0$ to $y_1 = h_1$. The quantity b_2 can be eliminated through the use of Equation (26). After carrying out the differentiation explicitly, the following expression is obtained:

$$l_1\frac{da_1}{dx_1^*} + l_2\frac{db_1}{dx_1^*} = l_3 \tag{27}$$

where

$$l_1 = -\frac{13}{10}\frac{1}{K_W} \tag{28a}$$

$$l_2 = 1 + \frac{13}{30}\frac{1}{K_W} \tag{28b}$$

$$l_3 = \frac{2}{K_W}(a_1 - b_1) \tag{28c}$$

On the other hand, the equation developed for stream 2 [Equation (16)] in the previous case remains valid. Solving Equations (27) and (16) simultaneously, the following working equations are obtained:

$$\frac{da_1}{dx_1^*} = \frac{l_3 g_2 + l_2 g_3}{l_1 g_2 - l_2 g_1} \tag{29}$$

$$\frac{db_1}{dx_1^*} = -\frac{l_1 g_3 + l_3 g_1}{l_1 g_2 - l_2 g_1} \tag{30}$$

The computation scheme for Case (B) begins in the same manner as of Case (A). First, the parameters K_N, K_W, and H are specified. A proper initial value of b_1 can be obtained from the asymptotic expression [Equation (23)]. With these initial values of a_1 and b_1, Equations (19) and (20) can be numerically integrated until $\Delta_1^+ = 1$. [This can be determined from Equation (14a) once the values of a_1 and b_1 are known.] After this, Equations (29) and (30) are used instead and the integration proceeds until $a_1 = 1$. The required relationship of $a_{1,0}$ vs L^* is thus obtained.

Generally speaking, it is difficult to predetermine whether one would have Case (A) or Case (B) once the system parameters and the outlet condition of stream 2 are arbitrarily assigned. In other words, with specified values of K_N, K_W, and H and assumed value of $a_{1,0}$, one could have either Case (A) or Case (B). This possible complication has been taken care of by developing a generalized computer program which switches from one type of computation to another as the physical situation demands.

Case (C)

The thermal boundary layer of stream 1 is developing throughout the entire length of the exchanger while that of stream 2 becomes fully developed somewhere within the exchanger. Case (C) is in effect equivalent to Case (B) by interchanging the two streams. However, for the sake of clarity as well as for facilitating the discussions of the subsequent cases, this case will be treated separately. Depending upon whether Δ_1^+ reaches unity or not, Case (C) consists of two sections which are:

Zone 1. $x_1^* < x_c^*$, $\Delta_2^+ = 1$, $\Delta_1^+ < 1$

Zone 2. $x_1^* > x_c^*$, $\Delta_2^+ \leq 1$, $\Delta_1^+ \leq 1$

For Zone 1, $\Delta_2^+ = 1$. Consequently, a different temperature profile for stream 2 is needed. This is given as

$$\theta_2 = a_1 + a_2\left[\frac{y_2}{h_2} - \frac{1}{3}\left(\frac{y_2}{h_2}\right)^3\right] \tag{31}$$

Like Equation (25), this expression has two unknown quantities, a_1 and a_2, and satisfies the boundary condition of perfect insulation at the outer wall. The interphase condition becomes

$$\frac{3}{2}K_N\frac{b_1}{\Delta_1^+} = a_2 = \frac{K_N}{K_W}(a_1 - b_1) \tag{32}$$

Substituting Equation (31) into Equation (1a) for $j = 2$ and carrying out the integration from $y_2 = 0$ to $y_2 = h_2$ and eliminating a_2 through the use of Equation (32), the following expression is obtained:

$$m_1 = \frac{da_1}{dx_1^*} + m_2\frac{db_1}{dx_1^*} = -m_3 \tag{33}$$

where

$$m_1 = 1 \tag{34a}$$

$$m_2 = -\frac{13 K_N}{30 K_W + 13 K_N} \tag{34b}$$

$$m_3 = \frac{-2(a_1 - b_1)}{K_W H\left(1 + \frac{13}{30}\frac{K_N}{K_W}\right)} \tag{34c}$$

On the other hand, Equation (15) is still applicable to stream 1. In solving Equations (33) and (15) explicitly, the following expressions are obtained:

$$\frac{da_1}{dx_1^+} = \frac{f_3 m_2 + f_2 m_3}{f_1 m_2 - f_2 m_1} \tag{35}$$

$$\frac{db_1}{dx_1^+} = -\frac{f_1 m_3 + f_3 m_1}{f_1 m_2 - f_2 m_1} \qquad (36)$$

These are the working equations for the region where $\Delta_2^+ = 1$ but $\Delta_1^+ \leqq 1$. For Zone 2, since the thermal boundary layer of both streams are developing (that is, $\Delta_1^+ \leqq 1$, $\Delta_2^+ \leqq 1$), the equations developed for Case (A) [Equations (15) and (16)] remain valid.

The basic difference between Cases (A) [or (B)] and Case (C) [as well as (D) and (E)] is that in the former cases, the thermal boundary layer of stream 2 remains developing throughout the entire length of the exchanger while for the latter, the thermal boundary layer has already become fully established prior to its exit from the exchanger. The question of which type of physical situation one faces is dependent upon the chosen value of $a_{1,0}$. If the value of $a_{1,0}$ satisfies the requirement of Equation (24), the situation would correspond to either Case (A) or Case (B), and the computation should be carried out as described previously. On the other hand, if the value of $a_{1,0}$ is so chosen that

$$a_{1,0} < \frac{1}{1 + \dfrac{2}{3}\dfrac{K_N}{K_W}} \qquad (37)$$

the situation could be either Case (C), (D) or (E).

The computation scheme can be briefly described as follows. A set of parameters K_N, K_W, *and* H is first selected. A proper value of $a_{1,0}$ [Equation (37)] is chosen. Based on this value of $a_{1,0}$, the initial value of b_1 can be obtained from Equation (23). With the initial value of a_1 and b_1 known, Equations (35) and (36) can be integrated numerically until

$$a_1 = \frac{1 + \dfrac{2}{3}\dfrac{K_N}{K_W} b_1}{1 + \dfrac{2}{3}\dfrac{K_N}{K_W}}$$

(or $\Delta_2^+ = 1$). Then Equations (19) and (20) may be used and the integration continued until $a_1 = 1$. It should be mentioned that the implicit assumption for this case is that $\Delta_1^+ \leqq 1$ throughout the entire exchanger length.

Case (D)

The exchanger consists of three zones which are:

Zone 1. $x_1^* < x_{c1}^*$, $\quad \Delta_1^+ < 1$ but $\Delta_2^+ = 1$

Zone 2. $x_{c1}^* < x_1^* < x_{c2}^*$ $\quad \Delta_1^+ \leqq 1$ and $\Delta_2^+ \leqq 1$

Zone 3. $x_1^* > x_{c2}^*$ $\quad \Delta_1^+ = 1$, but $\Delta_2^+ < 1$

For this case, the computation will begin with Equations (35) and (36) until

$$a_1 = \frac{1 + \dfrac{2}{3}\dfrac{K_N}{K_W} b_1}{1 + \dfrac{2}{3}\dfrac{K_N}{K_W}}$$

which marks the end of Zone 1. The computation will continue with Equations (19) and (20) until Δ_1^+ reaches unity. The calculation will then switch to Equations (29) and (30) until $a_1 = 1$.

Case (E)

The three zones are characterized as follows:

Zone 1. $x_1^* < x_{c1}^*$, $\Delta_1^+ < 1$ but $\Delta_2^+ = 1$

Zone 2. $x_{c1}^* < x_1^* < x_{c2}^*$, $\Delta_1^+ = 1$ and $\Delta_2^+ = 1$

Zone 3. $x_1^* > x_{c2}^*$, $\Delta_1^+ = 1$ but $\Delta_2^+ < 1$

For Zones 1 and 3, the computation procedure is the same as that described in the previous case, namely, Equations (35) and (36) for Zone 1 and Equations (29) and (30) for Zone 3. For Zone 2, both thermal boundary layers are fully developed. Consequently, Equation (27) for stream 1 and Equation (33) for stream 2 should be used. Explicitly, the working equations are given as follows:

$$\frac{da_1}{dx_1^*} = \frac{l_3 m_2 + l_2 m_3}{l_1 m_2 - l_2 m_1} \qquad (38)$$

$$\frac{db_1}{dx_1^*} = -\frac{l_1 m_3 + l_3 m_1}{l_1 m_2 - l_2 m_1} \qquad (39)$$

The computation scheme in this case proceeds as follows. For a set of parameters of K_N, K_W, and H, an assumed value of $a_{1,0}$ which obeys Equation (37), together with the initial value of b_1 [from Equation (23)] for an arbitrary small distance of x_1^*, Equations (35) and (36) will be integrated numerically until $\Delta_1^+ = 1$ (this can be calculated from Equation (32). Note that this occurs while the value of a_1 still remains less than

$$\frac{1 + \dfrac{2}{3}\dfrac{K_N}{K_W} b_1}{1 + \dfrac{2}{3}\dfrac{K_N}{K_W}}$$

The computation will switch to Equations (38) and (39) until

$$a_1 = \frac{1 + \dfrac{2}{3}\dfrac{K_N}{K_W} b_1}{1 + \dfrac{2}{3}\dfrac{K_N}{K_W}}$$

After that, Equations (29) and (30) will be used and integration proceeds until $a_1 = 1$ which yields the required value of L^* for the assumed initial value of $a_{1,0}$.

From the equations derived above, the primary information obtained is of the form of $a_{1,0}$ (or b_{1,L^*}) vs. L^* from which other pertinent heat transfer parameters can be obtained. The mixing cup mean temperature for either stream, by definition, is given as follows:

$$\theta_{j,B} = \frac{\int_o^{h_j} u_j \theta_j dy_j}{\int_o^{h_j} u_j dy_j} \qquad j = 1, 2$$

In applying Equation (40) to stream 2 at exit (that is, $x_1^* = 0$) for Cases (A) and (B), the integral in the numerical should be evaluated in two parts: from $y_2 = 0$, to $y_2 = \Delta_2$, with θ_2 given by Equation (9) and u_2 given by Equation (4*a*), and from $y_2 = \Delta_2$ to $y_2 = h_2$, with $\theta_2 = 1$. The relationship between a_1 and $(\theta_{2,B})_{x_1^*=0}$ is given as:

$$(\theta_{2,B})_{x_1^*=0} = 1 - (1 - a_1)\left[\frac{3}{5}\Delta_2^{+2} - \frac{1}{4}\Delta_2^{+3}\right] \qquad (41)$$

For Cases (C), (D), and (E), θ_2 is given by Equation (31). The following expression is obtained:

$$(\theta_{2,B})_{x_1^*=0} = a_1\left[1 + \frac{13}{30}\frac{K_N}{K_W}\right] \qquad (42)$$

The corresponding exit bulk temperature of stream 1, $\left[(\theta_{1,B})_{x_1^*=L^*}\right]$, can be obtained from simple energy balance as:

$$(\theta_{1,B})_{x_1^*=L^*} = H\left[1-(\theta_{2,B})_{x_1^*=0}\right] \quad (43)$$

The local Nusselt numbers for the two streams are:

$$N_{Nu1} = \left(\frac{h_1}{k_1}\right)\left[\frac{q}{(T_1)_{y_1=0}-T_{1,B}}\right] = \frac{a_1-b_1}{K_W(b_1-\theta_{1,B})} \quad (44a)$$

$$N_{Nu2} = \left(\frac{h_2}{k_2}\right)\left[\frac{q}{T_{2,B}-(T_2)_{y_2=0}}\right] = \frac{K_N}{K_W}\;\frac{a_1-b_1}{(\theta_{2,B}-a_1)} \quad (44b)$$

where q, the local heat flux, is assumed to be in the direction of y_1. The mixing cup mean temperature is given by Equation (40), with the use of the appropriate expression for the temperature profile.

Case (F)

The special case $K_w = 0$ corresponds to zero interphase resistance. The present approach may be slightly modified to include this case. Derivations are shown for Case (A). The interphase condition may be written as

$$-K_1\left(\frac{\partial\theta_1}{\partial y_1}\right)_0 = K_2\left(\frac{\partial\theta_2}{\partial y_2}\right)_0 \quad (45)$$

Since there is no interphase resistance, $a_1 = b_1$ all along the length of the heat exchanger. The profiles described in Equations (8) and (9) are valid with b_1 in Equation (8) replaced by a_1. By using these in Equation (45) and simplifying, one gets

$$a_1 = \frac{\Delta_1^+}{\Delta_1^+ + K_N\ \Delta_2^+} \quad (46)$$

It can be seen that when $\Delta_1^+ = 0$ (that is $x_1^* = 0$) $a_1 = 0$, and when $\Delta_2^+ = 0$ ($x_1^* = L^*$) $a_1 = 1$. Substituting for a_1 using Equation (46) in Equations (10) and (11) (note: $b_1 = a_1$) and carrying out the necessary differentiation and rearrangement, the following pair of first-order differential equations are obtained:

$$r_1\frac{d\Delta_1^+}{dx_1^*} + r_2\frac{d\Delta_2^+}{dx_1^*} = r_3 \quad (47)$$

$$s_1\frac{d\Delta_1^+}{dx_1^*} + s_2\frac{d\Delta_2^+}{dx_1^*} = s_3 \quad (48)$$

where

$$r_1 = \frac{\left(\frac{\Delta_1^{+2}}{10}-\frac{\Delta_1^{+3}}{24}\right)K_N\ \Delta_2^+}{(\Delta_1^+ + K_N\ \Delta_2^+)^2} + \frac{\left(\frac{\Delta_1^{+2}}{5}-\frac{\Delta_1^{+3}}{8}\right)}{\Delta_1^+ + K_N\ \Delta_2^+} \quad (49)$$

$$r_2 = -\frac{K_N\left(\frac{\Delta_1^{+3}}{10}-\frac{\Delta_1^{+4}}{24}\right)}{(\Delta_1^+ + K_N\ \Delta_2^+)^2} \quad (50)$$

$$r_3 = \frac{1}{2(\Delta_1^+ + K_N\ \Delta_2^+)} \quad (51)$$

$$s_1 = -\frac{K_N\ \Delta_2^+\left(\frac{\Delta_2^{+2}}{10}-\frac{\Delta_2^{+3}}{24}\right)}{(\Delta_1^+ + K_N\ \Delta_2^+)^2} \quad (52)$$

$$s_2 = \frac{K_N\ \Delta_1^+\left(\frac{\Delta_2^{+2}}{10}-\frac{\Delta_2^{+3}}{24}\right)}{(\Delta_1^+ + K_N\ \Delta_2^+)^2} + \left(1-\frac{\Delta_1^+}{\Delta_1^+ + K_N\ \Delta_2^+}\right)\left(\frac{\Delta_2^+}{5}-\frac{\Delta_2^{+2}}{8}\right) \quad (53)$$

$$s_3 = -\frac{1}{2}\ \frac{1}{\Delta_2^+}\ \frac{1}{K_N\cdot H}\left[1-\frac{\Delta_1^+}{\Delta_1^+ + K_N\ \Delta_2^+}\right] \quad (54)$$

From Equations (47) and (48) the following equations are obtained:

$$\frac{d\Delta_1^+}{dx_1^*} = \frac{r_1 s_2 - r_2 s_3}{r_1 s_2 - r_2 s_1} \quad (55)$$

$$\frac{d\Delta_2^+}{dx_1^*} = \frac{r_1 s_3 - r_3 s_1}{r_1 s_2 - r_2 s_1} \quad (56)$$

The asymptotic expression for Δ_1^+ for small x_1^* may be obtained by using Equation (46) in Equation (10) (with $b_1 = a_1$) and carrying out the necessary integration after neglecting terms involving Δ_1^{+4}. The result is,

$$\Delta_1^+ = 5^{1/3}\ x_1^{*1/3} \quad (57)$$

The computation proceeds as follows: First, the system parameters H and K_N are specified. The asymptotic value for small x_1^* is obtained from Equation (57). A value of Δ_2^+ is then assumed and Equations (55) and (56) are integrated using the Runge-Kutta scheme. The value of a_1 is readily computed, once Δ_1^+ and Δ_2^+ are known, using Equation (46). The integration proceeds till $a_1 = 1$. The value of L^* is thus obtained. The bulk mean temperature for stream 2, $(\theta_{2,B})_{x_1^*=0}$, may be readily shown to be

$$(\theta_{2,B})_{x_1^*=0} = 1 + 0.25\ \Delta_2^{+3} - 0.6\ \Delta_2^{+2} \quad (58)$$

RESULTS AND DISCUSSION

From the foregoing analysis, the heat exchanger length is found to be a function of the heat transfer requirement as well as three other parameters, K_N, K_W, H. The five cases considered cover all the possible physical situations which may occur in a countercurrent heat exchanger. On the other hand, in terms of actual computation, it is rather difficult to make a prior determination on the correct equations which are to be used merely on the basis of the assigned values of K_N, K_W, and H as well as the heat transfer requirement. Consequently, a general computer program for IBM system 360 was written which enables the automatic selection of the correct scheme as the computation proceeds. Runge-Kutta's method was used for the integration of the pertinent differential equations

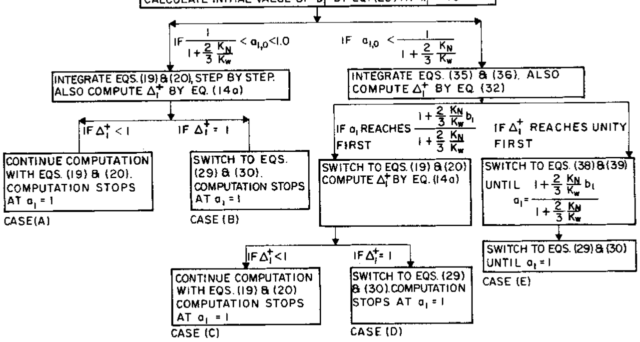

Fig. 3. Flow diagram of computation scheme.

[that is Equations (19) and (20), Equations (29) and (30), etc.] A flow diagram of the program is given in Figure 3.

TABLE 1.
$H = 1.125 \quad K_W = 2.0 \quad K_N = 1.33$

$(\theta_{2,B})_{x_1^*=0}$	$a_{1,0}$	L^*	$(\theta_{1,B})_{x_1^*=L^*}$
0.9983	0.92	0.00221	0.0019
0.9966	0.90	0.00451	0.00383
0.9881	0.85	0.0171	0.01338
0.9789	0.82	0.0321	0.02373
0.9708	0.80	0.0455	0.03285
0.9607	0.78	0.0632	0.04421
0.9486	0.76	0.0856	0.05782
0.9343	0.74	0.113	0.07391
0.9179	0.72	0.146	0.09236
0.911	0.7122	0.1605	0.1001
0.841	0.6528	0.32	0.179
0.531	0.4122	1.596	0.528

TABLE 2.
$H = 3.0 \quad K_N = 0.5 \quad K_W = 1.0$

$(\theta_{2,B})_{x_1^*=0}$	$a_{1,0}$	L^*	$(\theta_{1,B})_{x_1^*=L^*}$
0.9943	0.9	0.0116	0.0171
0.9900	0.88	0.02135	0.0300
0.9840	0.86	0.0359	0.048
0.9761	0.84	0.0567	0.072
0.9660	0.82	0.085	0.102
0.9535	0.80	0.1225	0.1395
0.9464	0.79	0.145	0.1608
0.941	0.7827	0.1605	0.177
0.896	0.7364	0.35	0.312
0.734	0.6033	1.7	0.798

TABLE 3.
$H = 3.0 \quad K_N = 0.5 \quad K_W = 0.1$

$(\theta_{2,B})_{x_1^*=0}$	$a_{1,0}$	L^*	$(\theta_{1,B})_{x_1^*=L^*}$
0.941	0.40	0.0556	0.177
0.908	0.35	0.109	0.276
0.873	0.315	0.176	0.381
0.808	0.269	0.354	0.577

$K_W = 0$

H	K_N	$(\theta_{2,B})_{x_1^*=0}$	L^* Ref. (7)	Present Work
1.125	1.33	0.939	0.015	0.0167
0.75	2.0	0.929	0.015	0.0167

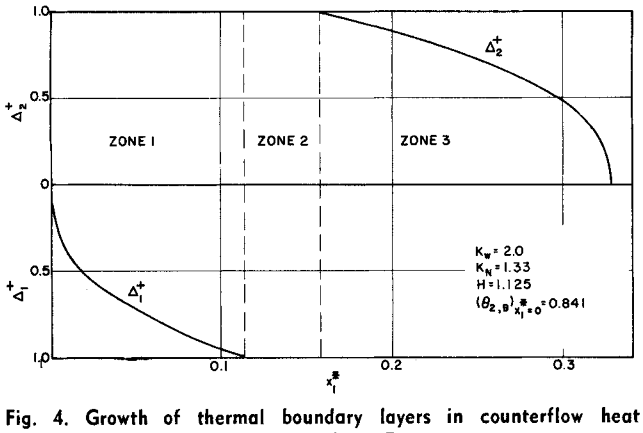

Fig. 4. Growth of thermal boundary layers in counterflow heat exchanger (case E).

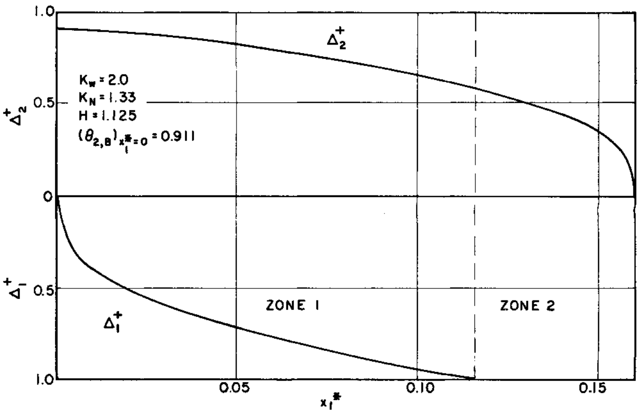

Fig. 5. Growth of thermal boundary layers in counterflow heat exchanger (case B).

Because of the large number of parameters involved, it is impractical to carry out an exhaustive calculation. Instead, only a limited number of cases were studied with a view to demonstrating the workability of the proposed method. The cases considered were:

1. $H = 1.125 \quad K_N = 1.33 \quad K_W = 2.0$
2. $H = 3.0 \quad K_N = 0.5 \quad K_W = 1.0$
3. $H = 3.0 \quad K_N = 0.5 \quad K_W = 1.0$
4. $H = 0.75$ and $1.125 \quad K_N = 2.0$ and $1.33 \quad K_W = 1.0$

For each case, the heat exchanger length was calculated for a given value of the exit bulk temperature, $(\theta_{2,B})_{x_1^*=0}$. These are summarized in Tables 1, 2, and 3. In Figure 4, the pattern of the thermal boundary layer growth for the special case of $K_W = 2.0$, $K_N = 1.33$, $H = 1.125$ and $(\theta_{2,B})_{x_1^*=0} = 0.841$ is given and illustrates the situation for Case (E) with all the three zones present. A similar plot depicting the situation for Case (B) is given in Figure 5.

A basic weakness of the integral method is the lack of a rigorous criterion which can be used to determine the accuracy of the results. Any inference of its relative accuracy can be obtained only through the comparison with results obtained from a more exact method. In Figures 6 and 7 and 8 the present result in the form of $(\theta_{2,B})_{x_1^*=0}$ vs. L^* are compared with those using orthogonal expansion technique (7) and reasonable agreement is observed in both cases. This plus the fact that integral solutions are found satisfactory in cocurrent exchanger (*14*) seems to indicate that integral approximation presents an effective alternative method for solutions of multistream exchangers.

It is also worth noting that the theoretical work of Mason (*5, 6*) which provide the mathematical foundation of the orthogonal expansion does not cover the case of plug flow because of its requirement of continuing the velocity functions throughout the domain of interest. Blanco (*1*) also mentioned the difficulty of carrying out the computation for the case of $H = 1$, although in a most recent publication (*13*), Stein presented a rigorous treatment for this special case. As far as can be seen, all these special cases do not present any difficulty to the integral method as discussed in this work.

The present work can also be modified to consider the effect of developing velocity profile in either (or both) of the streams. Because of the coupling of the momentum and energy equations, the orthogonal expansion technique which is based upon separation of variables would not

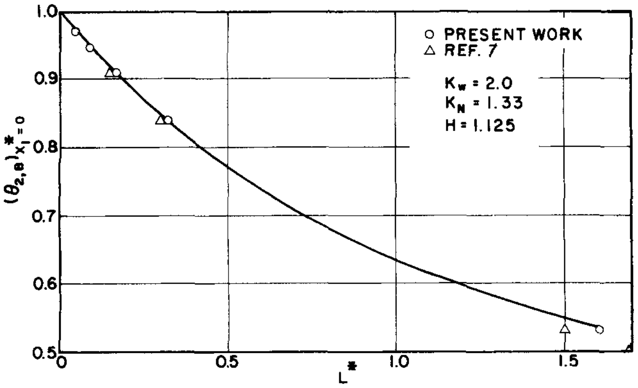

Fig. 6. Comparisons between integral solution and (7).

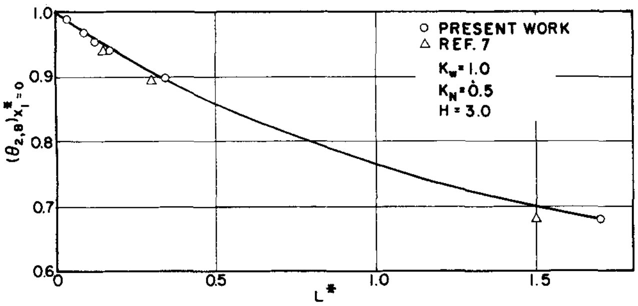

Fig. 7. Comparisons between integral solution and (7).

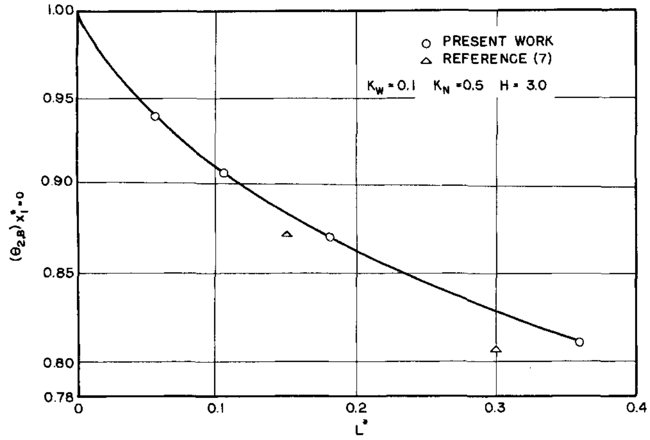

Fig. 8. Comparisons between integral solution and (7).

be applicable. With the use of the integral method, this would only mean the presence of two additional first-order differential equations. The calculation procedure probably would have to be iterative if the present approach is used. This is so because the exact location at which the velocity profile is fully established can only be determined by trial and error.

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NOTATION

a_1 = coefficient of temperature profile expression of stream 2, Equation (9)
$a_{1,0}$ = value of a_1 at $x_1^* = 0$
b_1 = coefficient of temperature profile expression of stream 1, Equation (8)
C_{pj} = heat capacity of stream j
f_1, f_2, f_3 = functions defined by Equations (17*a*), (17*b*), and (17*c*)
g_1, g_2, g_3 = functions defined by Equations (18*a*), (18*b*), and (18*c*)
H = parameter defined by Equation (12*a*)
h_j = width of passage of stream j
K_j = thermal conductivity of stream j
K = total conductance across interphase
K_N = parameter defined by Equation (12*b*)
K_W = parameter defined by Equation (13*a*)
L = total length of heat exchanger
L^* = dimensionless length, $L/h_1 N_{Pe1}$
l_1, l_2, l_3 = functions defined by Equations (28*a*), (28*b*), and (28*c*)
m_1, m_2, m_3 = functions defined by Equations (34*a*) (34*b*), and (34*c*)
N_{Nuj} = local Nusselt number of stream j defined by Equations (44*a*) and (44*b*)
N_{Pe1} = Peclet number for stream 1 defined by Equation (12*d*)
N_{Re1} = $2h_1\overline{U}_1\rho_1/\mu_1$
q = local heat flux
r_1, r_2, r_3 = functions defined by Equations (49), (50), and (51)
s_1, s_2, s_3 = functions defined by Equations (52), (53), and (54)
T_j = temperature of stream j
$T_{j,B}$ = cup mixing temperature of stream j
$(T_j)_i$ = inlet temperature of stream j
u_j = velocity of stream j
$\overline{U}_j$ = average velocity of stream j
W_j = mass velocity of stream j
x_j = longitudinal coordinate
x_1^* = dimensionless longitudinal coordinate defined by Equation (12*c*)
$x_c^*, x_{c1}^*, x_{c2}^*$ = dimensionless longitudinal distances when the thermal boundary layer of either stream has become fully developed, Figure 2
y_j = transverse coordinate

Greek Letters

α_j = thermal diffusivity of stream j
Δ_j = thermal boundary layer of stream j
Δ_j^+ = dimensionless thermal boundary layer thickness defined by Equation (12*e*)
θ_j = dimensionless temperature of stream j defined by Equation (5)
$\theta_{j,B}$ = dimensionless mixing cup mean temperature of stream j defined by Equation (40)
ρ_1 = density of stream 1

LITERATURE CITED

1. Blanco, Jose A., Ph.D. dissertation, Syracuse Univ., Syracuse, N.Y. (1967). Also, Blanco, J. A. and W. N. Gill, *Chem. Eng. Prog. Symp. Ser. No. 77,* **63**, 66 (1967).
2. ———, W. N. Gill, and R. J. Nunge, *AIChE J.*, **14**, 505 (1968).
3. King, C. J., *Ind. Eng. Chem. Fundamentals,* **4**, 125 (1965).
4. Gill, W. N., and A. J., Barduhn, *Chem. Eng.,* **16**, 167 (1962).
5. Mason, Max, *Trans. Am. Math. Soc.,* **7**, 337 (1906).
6. *Ibid.,* **8**, 427 (1907).
7. Nunge, R. J., Ph.D. dissertation, Syracuse Univ., Syracuse, N. Y. (1965).
8. ———, and W. N. Gill, *Int. J. Heat Mass Transfer,* **8**, 873 (1965).
9. ———, *AIChE J.,* **12**, 279 (1966).
10. Stein, R. P., in "Advances in Heat Transfer", T. F. Irvine and J. H. Hartnett, ed., Vol. III, pp. 101-174, Academic Press, New York (1966).
11. ———, in *proc. 3rd Intern. Heat Transfer Conference,* Vol. 1, p. 139, AIChE, New York (1966).
12. ———, *AIChE J.,* **12**, 1217 (1966).
13. ———, *AIChE,* Preprint 33 (1967).
14. Tien, C., "Proceedings of Symposium on Chemical Engineering in Medicine and Biology," p. 257, Plenum Press, New York (1967).

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